

PRESERVATION OF A SOUSLIN TREE AND SIDE CONDITIONS

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ABSTRACT. We show how to force, with finite conditions, the forcing axiom $\text{PFA}(T)$, a relativization of PFA to proper forcing notions preserving a given Souslin tree T . The proof uses a Neeman style iteration with generalized side conditions consisting of models of two types, and a preservation theorem for such iterations. The consistency of this axiom was previously known by the standard countable support iteration, using a preservation theorem due to Miyamoto.

INTRODUCTION

In this article, using the techniques introduced by Neeman in [2], we give a consistency proof of the Forcing Axiom for the class of proper forcings that preserve a Souslin tree T i.e. $\text{PFA}(T)$ ¹. The novelty of this proof is that $\text{PFA}(T)$ is forced with finite conditions, using a forcing that acts like an iteration. Indeed, the known consistency proofs for this axiom made use of a result of Miyamoto ([1]), who showed that the property “is proper and preserves every ω_1 -Souslin tree” is preserved under a countable support iteration of proper forcings.

The main preservation theorem presented here, Theorem 3.13, can be seen as a general preservation schema for properties, like being a Souslin tree, that have formulations similar to Lemma 1.2, in terms of the possibility to construct a generic condition for a product forcing, by means of conditions that, singularly, are generic for their respective forcings. As a matter of fact, in the proof of Theorem 3.13, no use is made of the fact that T is a tree.

In Section 1 we review some basic results connecting the property of being Souslin and properness. In Section 2 we show, as a warm up, that the method of side conditions - with just countable models - does not influence the fact that a proper forcing \mathbb{P} preserves a Souslin tree T . Then in Section 3 we use the method of generalized side conditions with models of two types to construct a model where $\text{PFA}(T)$ holds and T remains

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¹See [3], for a survey of interesting applications of $\text{PFA}(T)$.

Souslin. We refer to [2] and [4] for a detailed presentation of a pure side conditions poset with both countable and uncountable models.

1. SOUSLIN TREES AND PROPERNESS

We will use the following reformulation of the definition of Souslin tree.

Lemma 1.1. *A tree T is Souslin iff for every countable $M \prec H(\theta)$, with θ sufficiently large such that $T \in M$, and for every $t \in T_{\delta_M}$, where $\delta_M = M \cap \omega_1$,*

t is an (M, T) -generic condition,

i.e. for every maximal antichain $A \subseteq T$ in M , there is a $\xi < M \cap \omega_1$ such that $t \restriction \xi \in A$.

Proof. On the one hand, let T be a Souslin tree, $M \prec H(\theta)$ as above, $t \in T_{\delta_M}$ and $A \in M$ a maximal antichain of T . Since T is Souslin, A is countable. Then there is a $\alpha < \delta_M$ such that for all $\beta \geq \alpha$, the set $A \cap T_\beta$ is empty. Hence there is an element $h \in A$ compatible with $t \restriction \alpha$. Then $t \restriction ht(h) = s \in A$.

On the other hand if $A \in M$ is an uncountable maximal antichain of T , then $A \setminus M$ is not empty. For $x \in A \setminus M$, let $t = x \restriction \delta$. If there is a $\xi < \delta$ such that $t \restriction \xi \in A$, then x and $t \restriction \xi$ would be compatible and both in A : a contradiction. \square

The following lemma connects preservation of Souslin trees and properness.

Lemma 1.2. *(Miyamoto, Proposition 1.1 in [1]) Fix a Souslin tree T , a proper poset \mathbb{P} and some regular cardinal θ , large enough. Then the following are equivalent:*

- (1) $\Vdash_{\mathbb{P}}$ “ T is Souslin”,
- (2) *given $M \prec H(\theta)$ countable, containing \mathbb{P} and T , if $p \in \mathbb{P}$ is a (M, \mathbb{P}) -generic condition and $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$, then (p, t) is an $(M, \mathbb{P} \times T)$ -generic condition,*
- (3) *given $M \prec H(\theta)$ countable, containing \mathbb{P} and T and given $q \in \mathbb{P} \cap M$, there is a condition $p \leq q$ such that for every condition $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$, we have that (p, t) is an $(M, \mathbb{P} \times T)$ -generic condition.*

\square

2. PRESERVATION OF T AND COUNTABLE MODELS

We define the scaffolding operator from an idea of Veličković.

Definition 2.1. Given a proper poset \mathbb{P} and a sufficiently large cardinal θ such that $\mathbb{P} \in H(\theta)$, let $\mathbb{M}(\mathbb{P})$ be the poset consisting of conditions $p = (\mathcal{M}_p, w_p)$ such that

- (1) \mathcal{M}_p is a finite \in -chain of countable elementary substructures of $H(\theta)$,
- (2) $w_p \in \mathbb{P}$,
- (3) w_p is an (M, \mathbb{P}) -generic condition for every M in \mathcal{M}_p .

Moreover, we let $q \leq p$ iff $\mathcal{M}_p \subseteq \mathcal{M}_q$ and $w_q \leq_{\mathbb{P}} w_p$.

Remark 2.2. Notice that $\mathbb{M}(\mathbb{P})$ does not make reference to the cardinal θ . However this notation causes no confusion as long as θ depends on \mathbb{P} and its choice is a standard negligible part of all arguments involving properness. Then, without any specification, θ will always denote a cardinal that makes possible the definition of $\mathbb{M}(\mathbb{P})$.

Remark 2.3. By abuse of notation we will identify an \in -chain \mathcal{M}_p and the set of models that compose it.

Our aim now is to show that properness is preserved by the scaffolding operator.

Lemma 2.4. Let \mathbb{P} be a proper poset, $M \prec H(\theta)$ and $p \in \mathbb{M}(\mathbb{P}) \cap M$. Then there is a condition $p^M = (\mathcal{M}_{p^M}, w_{p^M}) \in \mathbb{M}(\mathbb{P})$ that is the largest condition extending p and such that $M \in \mathcal{M}_{p^M}$.

Proof. First of all notice that since $p \in M$, we have $\mathcal{M}_p \subseteq M$. In particular the largest model in \mathcal{M}_p belongs to M . So $\mathcal{M}_p \cup \{M\}$ is a finite \in -chain of elementary substructures of $H(\theta)$. Moreover $w_p \in M \cap \mathbb{P}$ and, by properness, there is a $w_q \leq w_p$ that is (M, \mathbb{P}) -generic. Now, since $w_q \leq w_p$ and w_p is (N, \mathbb{P}) -generic, for every $N \in \mathcal{M}_p$, so is w_q . Then we have that w_q is a generic condition for every model in $\mathcal{M}_p \cup \{M\}$. Finally set $\mathcal{M}_{p^M} = \mathcal{M}_p \cup \{M\}$ and $w_{p^M} = w_q$ to see that the conclusion of the lemma holds. \square

Theorem 2.5. Let \mathbb{P} be a proper poset. Then $\mathbb{M}(\mathbb{P})$ is proper.

Proof. Let M^* be a countable elementary submodel of $H(\theta^*)$, for some $\theta^* > \theta$, where θ is the corresponding cardinal in the definition of $\mathbb{M}(\mathbb{P})$. If p is a condition in $\mathbb{M}(\mathbb{P}) \cap M^*$ we need to find a condition $q \leq p$ that is $(M^*, \mathbb{M}(\mathbb{P}))$ -generic. Fix then a dense $D \subseteq \mathbb{M}(\mathbb{P})$ in M^* and let $M = M^* \cap H(\theta)$. We claim that $p^M = (\mathcal{M}_p \cup \{M\}, w_p^M)$ is an $(M, \mathbb{M}(\mathbb{P}))$ -generic condition.

Thanks to Lemma 2.4 we have that p^M is a condition. We now prove its genericity. Let $r \leq p^M$ and without loss of generality assume it to be in D . Define

$$E = \{w_s \in \mathbb{P} : \exists \mathcal{M}_s \text{ such that } (\mathcal{M}_s, w_s) \in D \wedge \mathcal{M}_r \cap M \subseteq \mathcal{M}_s\}$$

and notice that $E \in M^*$ and $w_r \in E$.

The set E may not be dense in \mathbb{P} , but

$$E_0 = \{w_t \in \mathbb{P} : \exists w_s \in E \text{ such that } w_t \leq w_s \text{ or } \forall w_s \in E (w_t \perp w_s)\}$$

is a dense subset of \mathbb{P} that belongs to M^* .

Then thanks to the (M^*, \mathbb{P}) -genericity of w_p^M and the fact that $w_r \leq w_p^M$, we have that there is a condition $w_t \in M^* \cap E_0$ that is compatible with w_r . Since $w_r \in E$ there is a condition $w_s \in E$ such that $w_t \leq w_s$. By elementarity can find w_s in M^* . Moreover, by definition of E , there is an \mathcal{M}_s such that $(\mathcal{M}_s, w_s) \in D$ and such that $\mathcal{M}_r \cap M \subseteq \mathcal{M}_s$. Again by elementarity we can find \mathcal{M}_s in M . Hence $(\mathcal{M}_s, w_s) \in D \cap M^*$.

Finally notice that w_s is compatible with w_r , because w_t is so and $w_t \leq w_s$; let w_a be the witness of it, i.e. $w_a \leq w_s, w_r$. Besides $\mathcal{M}_s \subseteq M$ and it extends $\mathcal{M}_r \cap M$, so we have that $\mathcal{M}_a = \mathcal{M}_s \cup \{M\} \cup \mathcal{M}_r \setminus M$ is a finite \in -chain of elementary submodel of $H(\theta)$. Then, in order to show that (\mathbb{M}_a, w_a) is a condition in $\mathbb{M}(\mathbb{P})$ we need to show that w_a is (N, \mathbb{P}) -generic, for every $N \in \mathcal{M}_a$. But this is true because on one hand $s \in \mathbb{M}(\mathbb{P})$ and so w_s is (N, \mathbb{P}) -generic for every $N \in \mathcal{M}_s$ and on the other hand $r \in \mathbb{M}(\mathbb{P})$ and so w_r is (N, \mathbb{P}) -generic for every $N \in \mathcal{M}_r$. Since w_a extends both w_s and w_r , we have that w_a is generic for all the models in \mathcal{M}_a . Hence a extends both s and r , in $\mathbb{M}(\mathbb{P})$, and witnesses their compatibility. \square

We now want to show that the scaffolding operation does not effect the preservation of a Souslin tree T . In order to show this fact we will use the characterization of Lemma 1.2.

Lemma 2.6. *Let T be a Souslin tree and let \mathbb{P} be a proper forcing, such that $\Vdash_{\mathbb{P}} "T \text{ is Souslin}"$. Moreover let M^* be a countable elementary submodel of $H(\theta^*)$, for some $\theta^* > \theta$, where θ is the corresponding cardinal in the definition of $\mathbb{M}(\mathbb{P})$. If $p \in \mathbb{M}(\mathbb{P})$, $M = M^* \cap H(\theta) \in \mathcal{M}_p$ and $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$, then (p, t) is an $(M^*, \mathbb{M}(\mathbb{P}) \times T)$ -generic condition.*

Proof. Fix a set $D \subseteq \mathbb{M}(\mathbb{P}) \times T$ dense in M^* and fix a condition $(r, t') \leq (p, t)$, that without loss of generality we can assume to be in D . Then define

$$E = \{(w_q, h) \in \mathbb{P} \times T \mid \exists \mathcal{M}_q \text{ such that } (q, h) \in D \text{ and } \mathcal{M}_r \cap M \subseteq \mathcal{M}_q\}$$

and notice that $E \in M$ and $(w_r, t') \in E$. Again the set E may not be dense, but the set $\bar{E} = E^{\leq} \cup E^{\perp}$, where

$$E^{\leq} = \{(w_s, u) \in \mathbb{P} \times T \mid \exists (w_q, h) \in E \text{ such that } (w_s, u) \leq (w_q, h)\} \text{ and}$$

$$E^{\perp} = \{(w_s, u) \in \mathbb{P} \times T \mid \forall (w_q, h) \in E (w_s, u) \perp (w_q, h)\},$$

is a dense subset of $\mathbb{P} \times T$ that belongs to M^* .

Now, since $M \in \mathcal{M}_r$, the condition w_r is (M, \mathbb{P}) -generic, by definition of $\mathbb{M}(\mathbb{P})$. Moreover since $\Vdash_{\mathbb{P}} "T \text{ is Souslin}"$ we have that (w_r, t') is $(M^*, \mathbb{P} \times T)$ -generic. Then there is a $(w_s, u) \in \bar{E} \cap M^*$, that is compatible with (w_r, t') . This latter fact then implies that $(w_s, u) \in E^{\leq} \cap M^*$ and so there is a condition $(w_q, h) \in E$ such that $(w_s, u) \leq (w_q, h)$. By elementarity we can find $(w_q, h) \in M^*$ and again, by elementarity we can assume $q = (\mathcal{M}_q, w_q)$ to be in M^* and so $(q, h) \in D \cap M^*$. Finally letting $\mathcal{M}_e = \mathcal{M}_q \cup \{M\} \cup \mathcal{M}_r \setminus M$, and w_e be the witness of the compatibility between w_q and w_r , we have that $e = (\mathcal{M}_e, w_e) \in \mathbb{M}(\mathbb{P})$ and that (e, t') extends both (r, t') and (q, h) . \square

Corollary 2.7. *Let T be a Souslin tree and let \mathbb{P} be a proper forcing. Then $\Vdash_{\mathbb{P}} "T \text{ is Souslin}"$ implies $\Vdash_{\mathbb{M}(\mathbb{P})} "T \text{ is Souslin}"$.* \square

3. PFA(T) WITH FINITE CONDITIONS

We now show that it is possible to force an analog of the Proper Forcing Axiom for proper poset that preserve a given Souslin tree T . We will follow Neeman's presentation of the consistency of PFA with finite conditions, from [2], arguing that a slightly modification of his method is enough for our purposes. Then we will argue that in the model we build T remains Souslin

Recall Neeman's definition of the forcing \mathbb{A} (Definition 6.1 from [2]). Fix a supercompact cardinal θ and a Laver function $F : \theta \rightarrow H(\theta)$ as a book-keeping for choosing the proper posets that preserve T . Moreover define Z as the set of ordinals α , such that $(H(\alpha), F \upharpoonright \alpha)$ is elementary in $(H(\theta), F)$. Then let $\mathcal{Z}^\theta = \mathcal{Z}_0^\theta \cup \mathcal{Z}_1^\theta$, where \mathcal{Z}_0^θ is the collection of all countable elementary substructure of $(H(\theta), F)$ and \mathcal{Z}_1^θ is the collection of all $H(\alpha)$, such that $\alpha \in Z$ has uncountable cofinality - hence $H(\alpha)$ is countably closed. Moreover, for $\alpha \in Z$, let $f(\alpha)$ be the least cardinal such that $F(\alpha) \in H(f(\alpha))$. Notice that, by elementarity, $f(\alpha)$ is smaller than the next element of Z above α .

Definition 3.1. *If \mathcal{M} is a set of models in \mathcal{Z}^θ , let $\pi_0(\mathcal{M}) = \mathcal{M} \cap \mathcal{Z}_0^\theta$ and $\pi_1(\mathcal{M}) = \mathcal{M} \cap \mathcal{Z}_1^\theta$.*

With an abuse of notation we will identify an \in -chain of models with the set of models that belong to it.

Definition 3.2. *Let \mathbb{M}_θ^2 the poset, whose conditions \mathcal{M}_p are \in -chains of models in \mathcal{Z}^θ , closed under intersection. If $p, q \in \mathbb{M}_\theta^2$, we define $p \leq q$ iff $\mathcal{M}_q \subseteq \mathcal{M}_p$.*

See Claim 4.1 in [2] for the proof that \mathbb{M}_θ^2 is \mathcal{Z}^θ -strongly proper.

Definition 3.3. *Conditions in the poset $\mathbb{A}(T)$ are pairs $p = (\mathcal{M}_p, w_p)$ so that:*

- (1) $\mathcal{M}_p \in \mathbb{M}_\theta^2$.
- (2) w_p is a partial function on θ , with domain contained in the (finite) set $\{\alpha < \theta : H(\alpha) \in p \text{ and } \Vdash_{\mathbb{A}(T) \cap H(\alpha)} "F(\alpha) \text{ is a proper poset, that preserves } T"\}$.
- (3) For $\alpha \in \text{dom}(w_p)$, $w_p(\alpha) \in H(f(\alpha))$.
- (4) $\Vdash_{\mathbb{A}(T) \cap H(\alpha)} w_p(\alpha) \in F(\alpha)$.
- (5) If $M \in \pi_0(\mathcal{M}_p)$ and $\alpha \in M$, then $(p \cap H(\alpha), w_p \upharpoonright \alpha) \Vdash_{\mathbb{A}(T) \cap H(\alpha)} "w_p(\alpha) \text{ is an } (M[\dot{G}_\alpha], F(\alpha))\text{-generic condition}"$, where \dot{G}_α is the canonical name for a generic filter on $\mathbb{A}(T) \cap H(\alpha)$.

The ordering on $\mathbb{A}(T)$ is the following: $q \leq p$ iff $\mathcal{M}_p \subseteq \mathcal{M}_q$ and for every $\alpha \in \text{dom}(w_p)$, $(\mathcal{M}_q \cap H(\alpha), w_q \upharpoonright \alpha) \Vdash_{\mathbb{A}(T) \cap H(\alpha)} "w_q(\alpha) \leq_{F(\alpha)} w_p(\alpha)"$.

Remark 3.4. This inductive definition makes sense, since $\mathbb{A}(T) \cap H(\alpha)$ is definable in any $M \in \mathcal{Z}_0^\theta$, with $\alpha \in M$.

Remark 3.5. Condition (5) holds for α and M iff it holds for α and $M \cap H(\gamma)$, whenever $\gamma \in Z \cup \{\theta\}$, is larger than α .

Definition 3.6. *Let β be an ordinal in $Z \cup \{\theta\}$. The poset $\mathbb{A}(T)_\beta$ consists of conditions $p \in \mathbb{A}(T)$ such that $\text{dom}(w_p) \subseteq \beta$.*

Remark 3.7. In order to simplify the notation, if $p \in \mathbb{A}(T)$, then we define $(p)_\alpha$ to be $(\mathcal{M}_p, w_p \upharpoonright \alpha)$, while by $p \upharpoonright H(\alpha)$ we denote $(\mathcal{M}_p \cap H(\alpha), w_p \upharpoonright \alpha)$. Notice that $(p)_\alpha \in \mathbb{A}(T)_\alpha$ and $p \upharpoonright H(\alpha) \in \mathbb{A}(T) \cap H(\alpha)$.

Following Neeman it is possible to prove the following facts. See [2] for their proofs in the case of the forcing \mathbb{A} i.e. the poset that forces PFA with finite conditions. Indeed, the only difference between \mathbb{A} and $\mathbb{A}(T)$ is that the Laver function F picks up a smaller class of proper posets; namely the class of proper poset that preserve T .

Theorem 3.8. *(Neeman, Lemma 6.7 in [2]) Let $\beta \in Z \cup \{\theta\}$. Then $\mathbb{A}(T)_\beta$ is \mathcal{Z}_1^θ -strongly proper. \square*

Claim 3.9. *(Neeman, Claim 6.10 in [2]) Let $p, q \in \mathbb{A}(T)$. Let $M \in \pi_0(\mathcal{M}_p)$ and suppose that $q \in M$. Suppose that for some $\delta < \theta$, p extends $(q)_\delta$ and $\text{dom}(w_q) \setminus \delta$ is disjoint from $\text{dom}(w_p)$. Suppose further that $(\mathcal{M}_p \cap M) \setminus H(\delta) \subseteq \mathcal{M}_q$. Then there is $w_{p'}$ extending w_p so that $\text{dom}(w_{p'}) = \text{dom}(w_p) \cup (\text{dom}(w_q) \setminus \delta)$ and so that $p' = (\mathcal{M}_p, w_{p'})$ is a condition in $\mathbb{A}(T)$ extending q . \square*

Theorem 3.10. *(Neeman, Lemma 6.11 in [2]) Let $\beta \in Z \cup \{\theta\}$. Let p be a condition in $\mathbb{A}(T)_\beta$. Let $\theta^* > \theta$ and let $M^* \prec H(\theta^*)$ be countable with $F, \beta \in M^*$. Let $M = M^* \cap H(\theta)$ and suppose that $M \in \pi_0(\mathcal{M}_p)$. Then:*

- (1) for every $D \in M^*$ which is dense in $\mathbb{A}(T)_\beta$, there is $q \in D \cap M^*$ which is compatible with p . Moreover there is $r \in \mathbb{A}(T)_\beta$ extending both p and q , so that $\mathcal{M}_r \cap M \setminus H(\beta) \subseteq \mathcal{M}_q$, and every model in $\pi_0(\mathcal{M}_r)$ above β and outside M are either models in \mathcal{M}_p or of the form $N' \cap W$, where N' is a model in $\pi_0(\mathcal{M}_p)$.
- (2) p is an $(M^*, \mathbb{A}(T)_\beta)$ -generic condition. \square

Theorem 3.11. (Neeman, Lemma 6.13 in [2]) After forcing with $\mathbb{A}(T)$, $PFA(T)$ holds. \square

In order to show that $\mathbb{A}(T)$ preserves T , we need the following claim.

Claim 3.12. If $\Vdash_{\mathbb{A}(T)_\alpha}$ “ T is Souslin”, then $\Vdash_{\mathbb{A}(T)_\alpha \cap H(\alpha)}$ “ T is Souslin”.

Proof. In order to show that $\mathbb{A}(T)_\alpha \cap H(\alpha)$ preserves T , we use the equivalent formulation of Claim 1.2. Then, fix a countable $M^* \prec H(\theta^*)$, with $\theta^* > \theta$ and $\alpha, T \in M^*$. Then, following Remark 3.4, both $\mathbb{A}(T)_\alpha \cap H(\alpha)$ and $\mathbb{A}(T)_\alpha$ are definable in M^* . If $p \in (\mathbb{A}(T)_\alpha \cap H(\alpha)) \cap M^*$, then we want to show that there is a condition $p' \leq p$ such that for every $t \in T_{\delta_{M^*}}$, with $\delta_{M^*} = M^* \cap \omega_1$, the condition (p', t) is $(M^*, (\mathbb{A}(T)_\alpha \cap H(\alpha)) \times T)$ -generic.

Let $M = M^* \cap H(\theta)$ and \mathcal{M}_{p^M} be the closure under intersection of $\mathcal{M}_p \cup \{M\}$. It is easy to check that it is possible to find a function w_{p^M} with the same domain of w_p such that $p^M = (\mathcal{M}_{p^M}, w_{p^M})$ is a condition in $\mathbb{A}(T)_\alpha$ and such that $p^M \restriction H(\alpha) \leq p$. We claim that $p^M \restriction H(\alpha)$ is the condition we need: i.e. $(p^M \restriction H(\alpha), t)$ is an $(M^*, (\mathbb{A}(T)_\alpha \cap H(\alpha)) \times T)$ -generic condition, for every $t \in T_{\delta_{M^*}}$.

To this aim fix a set $D \in M^*$ dense in $(\mathbb{A}(T)_\alpha \cap H(\alpha)) \times T$, let $t \in T_{\delta_{M^*}}$ and assume $(p^M \restriction H(\alpha), t) \in D$. By Theorem 3.10, p^M is an $(M^*, \mathbb{A}(T)_\alpha)$ -generic condition. Then, thanks to our hypothesis, (p^M, t) is an $(M, \mathbb{A}(T)_\alpha \times T)$ -generic condition.

Now define E to be the set of conditions $(q, h) \in \mathbb{A}(T)_\alpha \times T$ such that $(q \restriction H(\alpha), h) \in D$ and such that $\mathcal{M}_{p^M} \cap M \subseteq \mathcal{M}_q$. Notice that $(p^M, t) \in E$ and $E \in M^*$. The set E may not be dense, but $E_0 = E_0^\leq \cup E_0^\perp$, where

$$E_0^\leq = \{(q_0, h_0) : \exists (q, h) \in E \text{ such that } (q_0, h_0) \leq (q, h)\},$$

and

$$E_0^\perp = \{(q_0, h_0) : \forall (q, h) \in E \ (q_0, h_0) \perp (q, h)\},$$

is a dense subset of $\mathbb{A}(T)_\alpha \times T$ belonging to M^* .

Then there is $(q_0, h_0) \in E_0 \cap M^*$ that is compatible with (p^M, t) . Since $(p^M, t) \in E$, by definition of E_0 , there is a condition $(q, h) \in E$ that is compatible with (p^M, t) . By elementarity we can assume $(q, h) \in E \cap M^*$. Now, the key observation is that by strong genericity of the pure side conditions if (r, t) witnesses that (p^M, t) and (q, h) are compatible, then

$(r \restriction H(\alpha), t)$ witnesses that $(p \restriction H(\alpha), t)$ and $(q \restriction H(\alpha), h)$ are compatible. This is sufficient for our claim, because by definition of E and since q is finite, $(q \restriction H(\alpha), h) \in D \cap M^*$. \square

We can now state and prove the main preservation theorem of this section.

Theorem 3.13. *If G is a generic filter for $\mathbb{A}(T)$, then in $V[G]$ the tree T is Souslin.*

Proof. We proceed by induction on β , proving that $\mathbb{A}(T)_\beta$ preserves T . If β is the first element of Z , then $\mathbb{A}(T)_\beta = \mathbb{M}_\theta^2$.

Claim 3.14. *The forcing \mathbb{M}_θ^2 preserves T .*

Proof. Let $M^* \prec H(\theta^*)$ be a countable model with $\theta^* > \theta$, containing \mathbb{M}_θ^2 and T , and let $\mathcal{M}_p \in \mathbb{M}_\theta^2$ be an $(M^*, \mathbb{M}_\theta^2)$ -generic condition, with $M = M^* \cap H(\theta) \in \mathcal{M}_p$. Moreover, let $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$. Thanks to Lemma 1.2, it is sufficient to show that (\mathcal{M}_p, t) is an $(M^*, \mathbb{M}_\theta^2 \times T)$ -generic condition.

To this aim, let $D \in M^*$ be a dense subset of $\mathbb{M}_\theta^2 \times T$ and assume, by density of D , that $(\mathcal{M}_p, t) \in D$. Then define

$$E = \{h \in T : \exists \mathcal{M}_q \in \mathbb{M}_\theta^2 \text{ such that } (\mathcal{M}_q, h) \in D \wedge \mathcal{M}_p \cap M \subseteq \mathcal{M}_q\}.$$

Since $\mathbb{M}_\theta^2, D, \mathcal{M}_p \cap M \in M^*$, we have $E \in M^*$. The set E may not be dense in T but

$$\bar{E} = \{\bar{h} \in T : \exists h \in E (\bar{h} \leq h) \vee \forall h \in E (\bar{h} \perp h)\}.$$

belongs to M^* and it is dense in T .

By (M^*, T) -genericity of t , there is an $\bar{h} \in \bar{E} \cap M$ that is compatible with t . Moreover, since $(\mathcal{M}_p, t) \in D$, we have that $t \in E$. Since $t \in E$ and $\bar{h} \in \bar{E}$ are compatible, by definition of \bar{E} , there is $h \in E$, with $\bar{h} \leq h$. By elementarity pick such an h in M^* . Then, by definition of E , there is $\mathcal{M}_q \in \mathbb{M}_\theta^2$, with $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$, such that $(\mathcal{M}_q, h) \in D$. By elementarity we can find $\mathcal{M}_q \in M^*$. Then, since \mathcal{M}_p is (M, \mathbb{M}_θ^2) -strong generic and $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$, we have that \mathcal{M}_p and \mathcal{M}_q are compatible. Finally, t and \bar{h} are compatible because $t \leq \bar{h}$ and $\bar{h} \leq h$. Hence (\mathcal{M}_p, t) and (\mathcal{M}_q, h) are compatible in $\mathbb{M}_\theta^2 \times T$ and this compatibility, together with the fact that $(\mathcal{M}_q, h) \in D \cap M^*$, witnesses that (\mathcal{M}_p, t) is $(M^*, \mathbb{M}_\theta^2 \times T)$ -generic. \square

If β is the successor of α in Z , then, by inductive hypothesis $\mathbb{A}(T)_\alpha$ preserves T . In order to show that $\mathbb{A}(T)_\beta$ also preserves T , we use the characterization of Lemma 1.2. Then, let $M^* \prec H(\theta^*)$ be a countable model, with $\theta^* > \theta$, containing β, F and T . Notice that $\mathbb{A}(T)_\beta$ is definable

in M^* , with β as a parameter. Moreover let $p \in \mathbb{A}(T)_\beta$ be an $(M^*, \mathbb{A}(T)_\beta)$ -generic condition, with $M = M^* \cap H(\theta) \in \mathcal{M}_p$, and let $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$. Then we want to show that (p, t) is an $(M^*, \mathbb{A}(T)_\beta \times T)$ -generic condition.

By elementarity of M^* , $\alpha \in M^*$. Now, fix a V -generic filter G over $\mathbb{A}(T)_\alpha$, with $(p)_\alpha \in G$. By Theorem 3.10 $(p)_\alpha$ is an $(M^*, \mathbb{A}(T)_\alpha)$ -generic condition for M^* and so $M^*[G] \cap V = M^*$.

If $H(\alpha) \notin \mathcal{M}_p$ and p cannot be extended to a condition containing $H(\alpha)$, then $\mathbb{A}(T)_\beta$, below p , is equivalent to $\mathbb{A}(T)_\alpha$. Then, forcing below p , the conclusion follows by inductive hypothesis. Then, assume $H(\alpha) \in \mathcal{M}_p$.

Let $G_\alpha = G \cap H(\alpha)$. Then, by Theorem 3.8, we have that G_α is a V -generic filter on $\mathbb{A}(T) \cap H(\alpha)$, because $\mathbb{A}(T)_\alpha \cap H(\alpha) = \mathbb{A}(T) \cap H(\alpha)$. Without loss of generality, we can assume $\Vdash_{\mathbb{A}(T) \cap H(\alpha)} "F(\alpha) \text{ is a proper poset that preserves } T"$, because, otherwise $\mathbb{A}(T)_\beta$ is equal to $\mathbb{A}(T)_\alpha$ and again the conclusion follows by inductive hypothesis. Let $\mathbb{Q} = F(\alpha)[G_\alpha]$. Then, by properness of \mathbb{Q} in $V[G_\alpha]$, modulo extending p , we can assume $\alpha \in \text{dom}(w_p)$.

Fix $D \subseteq \mathbb{A}(T)_\beta \times T$ dense and in M^* . Without loss of generality assume $(p, t) \in D$. Since we will work in $V[G_\alpha]$, we need to ensure that $\Vdash_{\mathbb{A}(T) \cap H(\alpha)} "T \text{ is Souslin}"$. But this is true, by inductive hypothesis, as the Claim 3.12 shows.

Now, in $V[G_\alpha]$, define E to be the set of couples $(u, h) \in \mathbb{Q} \times T$ for which there is a condition $(q, h) \in \mathbb{A}(T)_\beta \times T$ such that

- (1) $w_q(\alpha)[G_\alpha] = u$,
- (2) $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$,
- (3) $(q, h) \in D$, and
- (4) $q \restriction H(\alpha) \in G_\alpha$.

Notice that $E \in M^*[G_\alpha]$ and that $(w_p(\alpha)[G_\alpha], t) \in E$. The set E may not be dense, but if we define $E_0 = E_0^\leq \cup E_0^\perp$, with

$$E_0^\leq = \{(u_0, h_0) \in \mathbb{Q} \times T : \exists (u, h) \in E (u_0, h_0) \leq (u, h)\}$$

and

$$E_0^\perp = \{(u_0, h_0) \in \mathbb{Q} \times T : \forall (u, h) \in E (u_0, h_0) \perp (u, h)\},$$

we have that E_0 is dense in $\mathbb{Q} \times T$. Moreover, notice that by elementarity E_0 is in $M^*[G_\alpha]$.

Now, since $M \in \pi_0(\mathcal{M}_p)$ and $\alpha \in M^* \cap H(\theta) = M$, we have that $\Vdash_{\mathbb{A}(T) \cap H(\alpha)} "w_p(\alpha) \text{ is an } (M^*[G_\alpha], F(\alpha))\text{-generic condition}"$, where \dot{G}_α is a $\mathbb{A}(T) \cap H(\alpha)$ -name for G_α . Moreover, $\Vdash_{\mathbb{A}(T) \cap H(\alpha)} "F(\alpha) \text{ is a proper poset that preserves } T"$ and, by inductive hypothesis and Lemma 3.12, $\Vdash_{\mathbb{A}(T) \cap H(\alpha)} "T \text{ is Souslin}"$. Then by Lemma 1.2 applied in $V[G_\alpha]$ we have that $(w_p(\alpha)[G_\alpha], t)$ is an $(M^*[G_\alpha], \mathbb{Q} \times T)$ -generic condition.

Hence, there is a condition $(u_0, h_0) \in E_0 \cap M^*[G_\alpha]$ that is compatible with $(w_p(\alpha)[G_\alpha], t)$. Moreover, since $(w_p(\alpha)[G_\alpha], t) \in E$ we have that $(u_0, h_0) \in E_0^{\leq}$. This means that there is $(u, h) \in E$ such that $(u_0, h_0) \leq (u, h)$. By construction (u, h) is compatible with $(w_p(\alpha)[G_\alpha], t)$ and by elementarity we can find such a condition in $M^*[G_\alpha]$. Let $u_\alpha \in \mathbb{Q}$ be a witness of the compatibility between $w_p(\alpha)[G_\alpha]$ and u . Notice that u_α is an $(N[G_\alpha], \mathbb{Q})$ -generic condition for all $N \in \pi_0(\mathcal{M}_p)$, with $\alpha \in N$, because $u_\alpha \leq w_p(\alpha)[G_\alpha]$. Since $(u, h) \in E$ there is a condition $q \in \mathbb{A}(T)_\beta$, with $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$ and $w_q(\alpha)[G_\alpha] = u$, such that $(q, h) \in D$. By elementarity let $q \in M^*[G_\alpha]$ and so $(q, h) \in M^*[G_\alpha] \cap D$. Since $M^*[G_\alpha] \subseteq M^*[G]$ and $M^*[G] \cap V = M^*$, we have $(q, h) \in D \cap M^*$. Now, by strong genericity of the pure side conditions, letting \mathcal{M}_r be the closure under intersection of $\mathcal{M}_p \cup \mathcal{M}_q$, we have that \mathcal{M}_r witnesses that \mathcal{M}_p and \mathcal{M}_q are compatible. Moreover every model in $\pi_0(\mathcal{M}_r)$ above β and outside M are either models in \mathcal{M}_p or of the form $N' \cap W$, where N' is a model in $\pi_0(\mathcal{M}_p)$ and $W \in \pi_1(\mathcal{M}_q)$. Then u_α is an $(N[G_\alpha], \mathbb{Q})$ -generic condition, for all $N \in \pi_0(\mathcal{M}_r)$, with $\alpha \in N$, because of Remark 3.5 together with the fact that u_α extends both $w_p(\alpha)[G_\alpha]$ and u .

Finally, back in V , let \dot{u} and \dot{u}_α be $\mathbb{A}(T)_\alpha \cap H(\alpha)$ -names for u and u_α . Moreover, let $e \in \mathbb{A}(T)_\alpha \cap H(\alpha)$ be sufficiently strong to force all the properties we showed for q , \dot{u} and \dot{u}_α . We can also assume that e extends both $q \restriction H(\alpha)$ and $p \restriction H(\alpha)$. Now notice that $\mathcal{M}_e \cup \mathcal{M}_r$ is already an \in -chain closed under intersection and so if $\mathcal{M}_s = \mathcal{M}_e \cup \mathcal{M}_r$ and $w_s = w_e \cup \{\alpha, \dot{u}_\alpha\}$, we have that s is a condition in $\mathbb{A}(T)_\beta$. Hence (s, t) witnesses that (p, t) and (q, h) are compatible.

If β is a limit point of Z , let again $M^* \prec H(\theta^*)$ be a countable model containing $\mathbb{A}(T)_\beta$ and F . Then if $p \in \mathbb{A}(T)_\beta$, with $M^* \cap H(\theta) = M \in \mathcal{M}_p$, and $t \in T_{\delta_M}$, with $\delta_M = M \cap \omega_1$, then, thanks to Lemma 1.2, it is sufficient to show that (p, t) is an $(M^*, \mathbb{A}(T)_\beta \times T)$ -generic condition, in order to prove that $\mathbb{A}(T)_\beta$ preserves that T is Souslin.

To this aim, let $\bar{\beta} = \sup(\beta \cap M^*)$ and let $\delta < \bar{\beta}$, in $Z \cap M^*$, be such that $\text{dom}(w_p) \subseteq \delta$. Moreover fix $D \in M^*$ dense in $\mathbb{A}(T)_\beta \times T$ and assume $(p, t) \in D$.

Now, define E as the set of conditions $((q)_\delta, h) \in \mathbb{A}(T)_\delta \times T$ that extend to conditions $(q, h) \in D$, with $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$. The set E belongs to M^* , but it may not be dense in $\mathbb{A}(T)_\delta \times T$. However the set $E_0 = E_0^{\leq} \cup E_0^\perp$ is dense in $\mathbb{A}(T)_\delta \times T$ and belongs to M^* ; where

$$E_0^{\leq} = \{(q_0, h_0) \in \mathbb{A}(T)_\delta \times T : \exists ((q)_\delta, h) \in E \text{ such that } (q_0, h_0) \leq ((q)_\delta, h)\},$$

and

$$E_0^\perp = \{(q_0, h_0) \in \mathbb{A}(T)_\delta \times T : \forall ((q)_\delta, h) \in E \ (q_0, h_0) \perp ((q)_\delta, h)\}.$$

Then, by the inductive hypothesis, find a condition $(q_0, h_0) \in E_0 \cap M^*$ that is compatible with $((p)_\delta, t)$. Moreover, since $((p)_\delta, t) \in E$ and it is compatible with (q_0, h_0) , we have that $(q_0, h_0) \in E_0^\leq$. Then, by definition of E_0^\leq , there is a condition $((q)_\delta, h) \in E$ such that $(q_0, h_0) \leq ((q)_\delta, h)$ and, so, that is compatible with $((p)_\delta, t)$. By elementarity pick such a condition in M^* . Moreover, thanks the fact that $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$ and that $\mathcal{M}_p \cap M$ witnesses the M -strong genericity of \mathcal{M}_p , we have that the compatibility between $((p)_\delta, t) = ((p)_\beta, t)$ and $((q)_\delta, h)$ is witnessed by a condition $((\mathcal{M}_r, w_1), t)$, where \mathcal{M}_r is the closure under intersection of $\mathcal{M}_p \cup \mathcal{M}_q$. Then we have that $\mathcal{M}_r \cap M \setminus H(\beta) \subseteq \mathcal{M}_q$, and that every model in $\pi_0(\mathcal{M}_r)$ above β and outside M are either models in \mathcal{M}_p or of the form $N' \cap W$, where N' is a model in $\pi_0(\mathcal{M}_p)$ and $W \in \pi_1(\mathcal{M}_q)$.

Now, let $(q, h) \in D$ witness that $((q)_\delta, h) \in E$. By elementarity, we can find $(q, h) \in D \cap M^*$. Then, thanks to the fact that $\mathcal{M}_r \cap M \setminus H(\beta) \subseteq \mathcal{M}_q$ we can apply Claim 3.9 and find a function w_2 , extending w_1 , defined as $\text{dom}(w_2) = \text{dom}(w_1) \cup (\text{dom}(w_q) \setminus \delta)$, such that $((\mathcal{M}_r, w_2), t)$ extends (q, h) . Setting $w_r = w_2 \cup w_p \restriction [\bar{\beta}, \beta)$, we claim that r belongs to $\mathbb{A}(T)_\beta$.

In order to show that this latter claim holds, it is sufficient to show that if $\alpha \in \text{dom}(w_p) \restriction [\bar{\beta}, \beta)$, then $p \restriction H(\alpha)$ forces that $w_r(\alpha) = w_p(\alpha)$ is an $(N[\dot{G}_\alpha], F(\alpha))$ -generic condition, where \dot{G}_α is the canonical name for a V -generic filter over $\mathbb{A}(T) \cap H(\alpha)$ and $N \in \pi_0(r)$, with $\alpha \in N$. Notice that $\alpha \in N$ implies $N \notin M$. Then, since p is a condition, the claim follows thanks to Remark 3.5 and the fact that every model in $\pi_0(\mathcal{M}_r)$ above β and outside M are either models in \mathcal{M}_p or of the form $N' \cap W$, where N' is a model in $\pi_0(\mathcal{M}_p)$.

Hence, finally we have that (r, t) belongs to $\mathbb{A}(T)_\beta \times T$ and that, by construction, it extends both (q, h) and (p, t) . \square

4. CONCLUSIONS

As stated in the introduction, Theorem 3.13 could be generalized to other forcings that admit a formulation similar to Lemma 1.2. Indeed the argument patterns of all new results of this paper are similar and in proving them we did not use essential properties of a tree T , except the characterization of Lemma 1.2. More formally, given a proper forcing \mathbb{P} we could define the following property for a forcing \mathbb{Q} .

$(*)_{[\mathbb{P}, \mathbb{Q}]}$: the forcing \mathbb{Q} is proper and if M is a countable elementary substructure of $H(\theta)$, for θ sufficiently large such that $\mathbb{P}, \mathbb{Q} \in M$, then if p is an (M, \mathbb{P}) -generic condition and q is an (M, \mathbb{Q}) -generic condition, then (p, q) is an $(M, \mathbb{P} \times \mathbb{Q})$ -condition.

Then Theorem 3.13 shows that $(*)_{[\mathbb{A}(T)_{\alpha}, T]}$ holds, for every $\alpha \in Z \cup \{\theta\}$. Notice that the forcing $\mathbb{A}(T)$ is not, properly speaking, an iteration. Hence it is not fully correct to say that the property “ T is Souslin” is preserved under finite support iteration. However, we think that understang the pure side conditions in terms of a real iteration would allow to extend the class of properties for which these preservation results hold.

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